10.1

(1) Pf: $\lim_{x\to 0}\frac{f(x)-f(-x)}{2x} = \frac{1}{2}\left(\lim_{x\to 0}\frac{f(x)-f(0)}{x} + \lim_{x\to 0}\frac{f(0)-f(-x)}{x}\right)$

$= \frac{1}{2}(f'(0)+f'(0)) = f'(0)$

(2) $\lim_{x\to 0}\frac{f(x)-f(-x)}{2x}$ exists $\Rightarrow$ f is differentiable at 0.

It's not true, take $f(x)=|x|$

10.3

Pf: $|f(x)-\varphi(x)| = |f(x)-f(a)-f'(a)(x-a)| = |x-a|\cdot\left|\frac{f(x)-f(a)}{x-a}-f'(a)\right|$.

$|f(x)-g(x)| = |f(x)-\mu-\lambda(x-a)| = |x-a|\cdot\left|\frac{f(x)-\mu}{x-a}-\lambda\right|$.

Then for any $|x-a|>0$,

$$|f(x)-g(x)|>|f(x)-\varphi(x)| \Leftrightarrow \left|\frac{f(x)-\mu}{x-a}-\lambda\right| > \left|\frac{f(x)-f(a)}{x-a}-f'(a)\right|.$$

By the definition of the limit, we only need to prove that

$$\lim_{x\to a}\left|\frac{f(x)-\mu}{x-a}-\lambda\right| > \lim_{x\to a}\left|\frac{f(x)-f(a)}{x-a}-f'(a)\right| = 0, \ \forall (\lambda,\mu)\neq(f'(a),f(a)).$$

f is differentiable at a.

Case 1: When $\mu\neq f(a)$, $\lim_{x\to a}\left|\frac{f(x)-\mu}{x-a}-\lambda\right| = +\infty > 0$

Case 2: When $\mu = f(a)$,

$$(\lambda,\mu)\neq(f'(a),f(a)) \Rightarrow \lambda\neq f'(a) \Rightarrow \lim_{x\to a}\left|\frac{f(x)-\mu}{x-a}-\lambda\right|$$

$$= \lim_{x\to a}\left|\frac{f(x)-f(a)}{x-a}-\lambda\right|$$

$$= |f'(a)-\lambda| > 0$$

□.

10.5 $f'(x) = \frac{f(x)-f(a)}{x-a} = 0 \Leftrightarrow f'(x)(x-a)-(f(x)-f(a)) = 0 \Leftrightarrow \left(\frac{f(x)-f(a)}{x-a}\right)' = 0$

Pf:

We define a function $g(x) = \begin{cases} \frac{f(x)}{x-a} & x\in]a,b] , \\ 0 & x=a \end{cases}$

It's obvious that $g(x)=\frac{f(x)}{x-a}$ is differentiable on $]a,b]$.

Since $\lim\limits_{x\to a^+}\frac{f(x)}{x-a} \overset{f(a)=0}{=\!=} \lim\limits_{x\to a^+}\frac{f(x)-f(a)}{x-a} = f'(a)=0$, g is continuous at a.

Thus $g\in C'(]a,b[)\cap C([a,b])$ and $g(a)=g(b)=0$, we have

$$g'(c)=0 \quad \text{for some } c\in\,]a,b[.$$

Since $g'(x)=\frac{f'(x)(x-a)-f(x)}{(x-a)^2} = \frac{f'(x)-\frac{f(x)-f(a)}{x-a}}{x-a}$, we have

$$f'(c)-\frac{f(c)-f(a)}{c-a}=0 \qquad \square.$$

10.7

Pf: Define $g(x)=\begin{cases}\frac{\sin x}{x} & x\in\,]0,\frac{\pi}{2}] \\ 1 & x=0\end{cases}$, then $g(x)\in C([0,\frac{\pi}{2}])$ by $\lim\limits_{x\to 0^+}\frac{\sin x}{x}=1$.

$\forall x\in\,]0,\frac{\pi}{2}]$, $g'(x)=\frac{\cos x\cdot x-\sin x}{x^2} = \frac{x-\tan x}{x^2}$.

Since $(x-\tan x)' = 1-(1+\tan^2 x) = -\tan^2 x\le 0 \Rightarrow x-\tan x\le 0-\tan 0=0$, $\forall x\in\,]0,\frac{\pi}{2}]$.

We have $g'(x)\le 0$, $\forall x\in\,]0,\frac{\pi}{2}]$, then

$$\frac{\pi}{2}=g(1)\le g(x)\le g(0)=1. \qquad \square$$

10.9. Here $I=[a,b]$.

Pf: We may assume that $f(b)\ge f(a)$. Otherwise, replace f by $-f$.

Thus we only need to prove that $f(b)-f(a)\le g(b)-g(a)$ i.e. $f(b)-g(b)\le f(a)-g(a)$.

Since $(f(x)-g(x))'\le |f'(x)|-g'(x)\le 0$ on $]a,b[$, we have

$$f(b)-g(b)\le f(a)-g(b). \qquad \square$$

10.11. (罗钩尺)

Pf:

$$f'(x)=\left(\frac{1}{(1-x^2)^{\frac{1}{2}}}\right)' = \frac{x}{(1-x^2)^{\frac{3}{2}}} = (f(x))^3\cdot x\ge 0,\ \forall x\ge 0.$$

Now we prove the consequence by induction on $n\in\mathbb{N}$.

If it holds for $n \le k$, then

$$f^{(k+1)}(x) = (f'(x))^{(k)}$$
$$= (f^3(x) \cdot x)^{(k)}$$
$$= (f^3(x))^{(k-1)} \cdot x + (f^3(x))^{(k)} \cdot x$$
$$\geq 0$$

$\square$.

Hence $f'(x) \geq 0 \Rightarrow f^{(n)}(x) \geq 0,\ \forall n \in \mathbb{N}$.

10.13 $\forall x \in I,\ \exists \xi \in I$ s.t.

$$f^{(n)}(\xi) - \frac{n!\,f(x)}{(x-x_1)\cdots(x-x_n)} = 0$$

Pf: If $x = x_i$ for some $1 \le i \le n$, it's nothing to prove. Here we always assume that

$$x \in I \setminus \{x_1, \cdots, x_n\}.$$

We define $g(y) \in C^n(I)$ s.t.

$$g(y) = f(y) - \frac{f(x)}{(x-x_1)\cdots(x-x_n)}(y-x_1)\cdots(y-x_n).$$

Then $g(x) = g(x_1) = \cdots = g(x_n) = 0$.

Let $\{x, x_1, \cdots, x_n\} = \{\tilde{x}_1, \cdots, \tilde{x}_{n+1}\}$ s.t. $\tilde{x}_1 < \cdots < \tilde{x}_n$.

$g(\tilde{x}_1) = \cdots = g(\tilde{x}_{n+1}) = 0 \Rightarrow \exists\, y_k \in]\tilde{x}_k, \tilde{x}_{k+1}[,\ 1 \le k \le n$ s.t.

$$g'(y_1) = \cdots = g'(y_n) = 0 \text{ and } y_1 < \cdots < y_n.$$

$\Rightarrow \exists\, z_k \in]y_k, y_{k+1}[,\ 1 \le k \le n-1$ s.t.

$$g^{(2)}(z_1) = \cdots = g^{(2)}(z_{n-1}) = 0$$

We repeat the above process and finally get a number $\xi \in I$ s.t.

$$g^{(n)}(\xi) = 0.$$

$$0 = g^{(n)}(\xi) = f^{(n)}(\xi) - \frac{f(x) \cdot n!}{(x-x_1)\cdots(x-x_n)}.$$ $\square$.

10.15.

Pf: Define $f(x) = y^2(x) + \frac{(y'(x))^2}{q(x)}$, $x \in [A, +\infty[$, then

$$f'(x) = 2y(x)\cdot y'(x) + \frac{2y'(x)y''(x)\cdot q(x) - (y'(x))^2\cdot q'(x)}{(q(x))^2}$$

$$\overset{y''=-q\cdot y}{=} 2y(x)\cdot y'(x) + \frac{-2y'(x)\cdot y(x)\cdot (q(x))^2 - (y'(x))^2\cdot q'(x)}{(q(x))^2}$$

$$= -\frac{(y'(x))^2\cdot q'(x)}{(q(x))^2} \leq 0, \quad \forall x \in [A, +\infty[$$

$\forall x \in [A, +\infty[$,

$$y^2(x) \leq f(x) \leq f(A) \text{ is finite.}$$

$$\Rightarrow \quad |y(x)| \leq \sqrt{f(A)}, \quad \forall x \geq A. \qquad \square$$

10.17.

Pf:

(1) $$\lim_{n\to+\infty}\left|\frac{u_{n+1}-c}{u_n-c}\right| = \lim_{n\to\infty}\left|\frac{f(u_n)-f(c)}{u_n-c}\right| \overset{u_n\to c}{=\!=} \lim_{x\to c}\left|\frac{f(x)-f(c)}{x-c}\right| = |f'(c)| > 1$$

$\Rightarrow \exists N_0 \in \mathbb{N}, \forall n \geq N$ s.t.

$$\left|\frac{u_{n+1}-c}{u_n-c}\right| \geq \frac{|f'(c)|+1}{2} > 1 \Rightarrow |u_{n+1}-c| > |u_n-c| \cdots > |u_{N_0}-c|$$

$\Rightarrow \lim_{n\to\infty}|u_n-c| > |u_{N_0}-c| > 0$ which is a contradiction.

(2) pf:

$\lim_{n\to\infty} u_n = c \Rightarrow \exists n_0 \in \mathbb{N}$ s.t. $u_{n_0} = c$, otherwise $u_n \neq 0$ $\forall n \in \mathbb{N}$, this is a contradiction

$$\Rightarrow u_n = f^{n-n_0}(u_{n_0}) = f^{n-n_0}(c) = c, \ \forall n \geq n_0.$$

Converse is obvious.

(3). pf:

Define $f(x) = 4x(1-x)$, $f(x) = x \Leftrightarrow x = 0$ or $\frac{3}{4}$.

Since $|f'(0)| = 4 > 1$, $|f'(\frac{3}{4})| = |4-8x|_{x=\frac{3}{4}} = 2 > 1$, we only need to prove that $u_n \neq 0$ & $u_n \neq \frac{3}{4}$, $\forall n \in \mathbb{N}$.

$0 < f(x) < 1,\ \forall x \in (0,1) \xRightarrow{0<u_0<1} 0 < u_n < 1, \forall n \in \mathbb{N} \Rightarrow u_n \neq 0, \forall n \in \mathbb{N}$.

It's easy to see that $u_n \in \mathbb{Q}$, $\forall n \in \mathbb{N}$ by induction.

If $u_{n_0} = \frac{3}{4}$ for some $n_0 \in \mathbb{N}$ s.t. $u_n \neq \frac{3}{4}$, $\forall n < n_0$, then

$$u_{n_0-1} = \frac{1}{4}$$

$$\Rightarrow 4u_{n_0-2}(1-u_{n_0-2}) = \frac{1}{4} \Rightarrow u_{n_0-2} = \frac{2-\sqrt{3}}{4}$$

Which is a contradiction

□.

10.19

pf: (1).

$$x \in [\tfrac{\pi}{4}, 1] \Rightarrow 2x \in [\tfrac{\pi}{2}, 2] \subseteq [\tfrac{\pi}{2}, \pi]$$

$$\Rightarrow 1 \geq \sin 2x \geq \sin 2 > \frac{\pi}{4}$$

(2)

Case 1: $u_{n_0} \in [\frac{\pi}{4}, 1]$ for some $n_0 \in \mathbb{N}$, WLOG, we may assume that $n_0 = 1$.

It's easy to see that $u_n \in [\frac{\pi}{4}, 1]$, $\forall n \geq 1$ by (1).

Since $(\sin 2x)' = 2\cos 2x$, then

$$-1 < 2\cos 2 \leq 2\cos 2x \leq 2\cos(2\cdot\tfrac{\pi}{4}) = 0,\ \forall x \in [\tfrac{\pi}{4}, 1]$$

$\Rightarrow |u_{n+1} - u_n| \leq C \cdot |u_n - u_{n-1}|$ for some constant $0 < C < 1$.

$$\Rightarrow |u_{n+1} - u_n| \leq C^n |u_1 - u_0|$$

$$\Rightarrow |u_{n+p} - u_n| \leq \sum_{k=0}^{p-1} |u_{n+k+1} - u_{n+k}|$$

$$\leq \sum_{k=0}^{p-1} C^{n+k} |u_1 - u_0|$$

$$= \frac{C^n(1-C^{p-1})}{1-C} |u_1 - u_0| < \frac{C^n}{1-C} |u_1 - u_0|$$

$\Rightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0, k \in \mathbb{N}$ s.t.

$$|u_{n+p} - u_n| < \varepsilon$$

$\Rightarrow \{u_n\}$ is a cauchy sequence

$\Rightarrow u_n \to u_0$ for some $u_0 \in [\frac{\pi}{4}, 1]$ s.t. $\sin 2u_0 = u_0$.

Case 2: $u_n \notin [\frac{\pi}{4}, 1]$, $\forall n \in \mathbb{N}$

$u_0 \in [0, \frac{\pi}{2}] \Rightarrow u_1 \in [0,1] \Rightarrow u_1 \in [0, \frac{\pi}{4}]$.

$\Rightarrow u_2 \in [0,1] \Rightarrow u_2 \in [0, \frac{\pi}{4}]$

....

$\Rightarrow u_n \in [0, \frac{\pi}{4}]$, $\forall n \in \mathbb{N}$.

Define $f(x) = \sin 2x - x$, $x \in [0, \frac{\pi}{4}]$, then $f'(x) = 2\cos 2x - 1$.

$\Rightarrow f'(x) > 0$ on $[0, \frac{\pi}{6})$ and $f'(x) < 0$ on $(\frac{\pi}{6}, \frac{\pi}{4}]$.

$\Rightarrow f(x) \nearrow$ on $[0, \frac{\pi}{6})$ and $f(x) \downarrow$ on $(\frac{\pi}{6}, \frac{\pi}{4}]$.

Since $f(0) = 0$ and $f(\frac{\pi}{4}) = 1 - \frac{\pi}{4} > 0$, we have

$f(x) \geq 0$.

and $f(x) = 0 \Leftrightarrow x = 0$.

$\Rightarrow u_{n+1} - u_n = \sin(2u_n) - u_n \geq 0$

$\Rightarrow u_n \uparrow$ & u_n is bounded from above by $\frac{\pi}{4}$

$\Rightarrow \exists a_0 \in [0, \frac{\pi}{4}]$ s.t. $u_n \to a_0$

$\Rightarrow a_0 = \lim_{n\to\infty} u_n = \lim_{n\to\infty} \sin 2u_n = \sin 2a_0$

If $u_1 \neq 0$, then $a_0 = \lim_{n\to\infty} u_n > u_1 > 0$ which contradicts to the above fact

$\sin 2x - x = 0$ on $[0, \frac{\pi}{4}] \Leftrightarrow x = 0$

$\Rightarrow u_1 = 0 \Rightarrow u_0 = 0$ or $\frac{\pi}{2}$.

In summary, when $u_0 = 0$ or $\frac{\pi}{2}$, $u_n = 0$, $\forall n \geq 1$.

when $u_0 \neq 0, \frac{\pi}{2}$, $u_n \to C$ for some constant c s.t. $\sin 2C = C$ & $C \in [\frac{\pi}{4}, 1]$